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The Trace of the Zygmund Class $\Lambda_k(R)$ to Closed Sets and Interpolating Polynomials*

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0. INTRODUCTION

Let $\Lambda_k(R^n)$ denote the Zygmund class of continuous functions satisfying $|\Delta_h^{k+1}f(x)| \leqslant c\,|h|^k$ and $|f(x)| \leqslant c$, $x, h \in R^n$, where $\Delta_h^{k+1}f(x)$ is the difference of order $k+1$ with step h at x (Definition 1 below). In [1], with further development in [2] (see also [5]), the author together with H. Wallin characterized the trace $\{f \mid F; f \in \Lambda_k(R^n)\}$ of $\Lambda_k(R^n)$ to an arbitrary closed set $F \subset R^n$ by means of certain approximation properties (Theorem 1 below). The characterizations given in those papers, however, are non-constructive. One consequence of this is that the extension theorem which is the main part of Theorem 1 is not obtained by means of a linear extension operator.

In this paper we give a characterization of the trace of $\Lambda_k(R)$ to an arbitrary closed subset F of R, which is of a more constructive character, using interpolating polynomials (Definition 3 and Theorem 2); observe that we now work in one dimension. As a consequence, we obtain a bounded, linear extension operator from the trace space to $\Lambda_k(R)$ (Theorem 3). In Proposition 1, we give a characterization of $\Lambda_k(R)$ by means of interpolating polynomials, which we have previously seen for $k=1$ only.

The problem studied in this paper is similar to one studied by H. Whitney in [7], where a characterization of the trace of $C^m(R)$, the class of m times continuously differentiable functions, to closed subsets of R was given with the aid of interpolating polynomials. Whitney's result has been generalized by J. Merrien in [3]. One could say that the present paper is related to [1] in the same way as Whitney's paper [7] is related to his paper [6], where he proves his classical extension theorem. To characterize the trace of the Lipschitz spaces $\Lambda_\alpha(R)$, α non-integer (cf. [4], Chap. VI), is

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a problem which is very close to the one studied in [3] and [7] and hence we consider the integer case only.

It is an open problem to obtain results similar to those in this paper, and to those in [7], in several dimensions.

1. Definitions and Results

We first give one of many possible definitions of the classical Zygmund (or Lipschitz) space $\Lambda_k(R^n)$. It is important to note that we deal with the spaces which are defined by means of a difference of order $k+1$ and not of order k, which leads to a different space (see, e.g., [4, p. 141]). By $\Delta_h^m f(x)$, $x, h \in R^n$, m positive integer, we denote the difference of order m with step h at the point x, i.e., $\Delta_h^1 f(x) = f(x+h) - f(x)$ and, for $m > 1$, $\Delta_h^m f(x) = \Delta_h^1(\Delta_h^{m-1} f)(x)$.

Definition 1 (the spaces $\Lambda_k(R^n)$). Let f be defined on R^n. Then $f \in \Lambda_k(R^n)$, k positive integer, if and only if f is continuous and for $x, h \in R^n$ and for some constant M satisfies

$$|\Delta_h^{k+1} f(x)| \leqslant M\,|h|^k \qquad \text{and} \qquad |f(x)| \leqslant M. \tag{1.1}$$

The *norm* of f in $\Lambda_k(R^n)$ is defined as the infimum of the constants M in (1.1). Alternatively, $\Lambda_k(R^n)$ may be defined as the space consisting of all functions f which have continuous and bounded derivatives up to order $k-1$, and with derivatives $f^{(j)}$ of order $k-1$ satisfying $|\Delta_h^2 f^{(j)}(x)| \leqslant M\,|h|$. Often spaces $\Lambda_\alpha(R^n)$ are defined for $\alpha > 0$, but since our interest lies in the integer case (cf. the introduction) we restrict ourselves to $\Lambda_k(R^n)$. Also, Definition 2 and Theorem 1 were in [2] given for spaces $\Lambda_\alpha(F)$, $\alpha > 0$, and also in a more general form from other aspects.

From now on, F will always denote a closed subset of R^n.

The spaces $\Lambda_k(F)$ in the next definition were given in [2, Theorem 1.3], see also [1, Remark 4.5]. In [2] we used a different notation. If $F = R^n$, the space $\Lambda_k(F)$ given by Definition 2 coincides with the space $\Lambda_k(R^n)$ in Definition 1, cf. [2, Proposition 1.3].

Definition 2 (the space $\Lambda_k(F)$). Let f be defined on F. Then $f \in \Lambda_k(F)$, k positive integer, if and only if the following condition holds: for every closed cube Q in R^n with $Q \cap F \neq \varnothing$ and with sides of length $\delta > 0$, there exists a polynomial P_Q of degree $\leqslant k$ such that

$$|f(x) - P_Q(x)| \leqslant c\delta^k, \qquad x \in Q \cap F; \tag{1.2}$$

if Q' is a cube with sides of length $\delta' > 0$, $Q' \cap F \neq \emptyset$, then

$$|P_Q(x) - P_{Q'}(x)| \leqslant c(\max(\delta, \delta'))^k, \qquad x \in Q \cap Q', \tag{1.3}$$

and

$$|P_Q(x)| \leqslant c, \qquad x \in Q, \quad \text{if } \delta = 1. \tag{1.4}$$

The *norm* of $f \in \Lambda_k(F)$ is the infimum of the constants c, such that (1.2)–(1.4) hold for some collection $\{P_Q\}$.

The interest in the spaces $\Lambda_k(F)$ comes from the following theorem, which says that $\Lambda_k(F)$ is the "trace" of the classical space $\Lambda_k(R^n)$ to F. By $f \mid F$ we denote the pointwise restriction to F of a continuous function f defined on R^n.

THEOREM 1. $\Lambda_k(F) = \{f \mid F; f \in \Lambda_k(R^n)\}$.

Theorem 1 is essentially an extension theorem, since if $f \in \Lambda_k(R^n)$, then it is obvious from Definition 2 that $f \mid F \in \Lambda_k(F)$. The interesting part of Theorem 1 is the fact that if $g \in \Lambda_k(F)$, then there exists an extension f defined on R^n of g with $f \in \Lambda_k(R^n)$; it is also known that the $\Lambda_k(R^n)$-norm of f is less than a constant times the $\Lambda_k(F)$-norm of g, where the constant only depends on k and n. Theorem 1 was given in [2] (see Corollary 1.1 of that paper). However, that corollary is in its turn more or less a reformulation of the extension theorem in [1].

We now come to the results of this paper, and work from now on in one dimension. In the following definition k is a positive integer, a denotes a set consisting of $k+1$ distinct points from F, $a = \{a_0, a_1, ..., a_k\}$, $|a|$ is the diameter of the set a, Q_a is the smallest closed interval containing a, and P_a is the unique polynomial of degree $\leqslant k$ interpolating f at $a_0, a_1, ..., a_k$. The letter b is used analogously. Also, if $0 \leqslant \nu < k$, $a(\nu)$ is a set of $\nu + 1$ distinct points from F, $P_{a(\nu)}$ is the polynomial of degree $\leqslant \nu$ interpolating f at the points of $a(\nu)$, and $Q_{a(\nu)}$ is the smallest interval containing $a(\nu)$.

DEFINITION 3 (the space $\Lambda_k^*(F)$). Let $F \subset R$ and let f be defined on F. Then $f \in \Lambda_k^*(F)$ if and only if to any a and b with k points in common and $|b| \geqslant |a|$

$$|P_a^{(k)} - P_b^{(k)}| \leqslant c(1 + \ln(|b|/|a|)), \tag{1.5}$$

and for any a and any $a(\nu)$, $0 \leqslant \nu < k$,

$$|P_{a(\nu)}^{(\nu)}| \leqslant c \qquad \text{and} \qquad |P_a^{(k)}| \leqslant c(1 + \max(\ln(1/|a|), 0)). \tag{1.6}$$

The *norm* of $f \in \Lambda_k^*(F)$ is the infimum of the constants c appearing in (1.5) and (1.6).

Here $P^{(\nu)}_{a(\nu)}$ and $P^{(k)}_a$ denote derivatives of order ν and k, respectively, so they are constant functions.

The next theorem is proved in Section 3.

THEOREM 2. *Let $F \subset R$. Then*

$$\Lambda^*_k(F) = \Lambda_k(F)$$

with equivalent norms. More precisely, there exist constants c_1 and c_2, depending only on k, such that, if $f \in \Lambda_k(F)$ or $f \in \Lambda^*_k(F)$,

$$c_1 \|f\|_{\Lambda_k(F)} \leqslant \|f\|_{\Lambda^*_k(F)} \leqslant c_2 \|f\|_{\Lambda_k(F)}. \tag{1.7}$$

Thus, by Theorem 1, $\Lambda^*_k(F)$ is an alternative characterization of the trace of $\Lambda_k(R^n)$ to F for the case $n = 1$. It has the advantage of being of a more constructive character, which leads to the following theorem (see the end of Section 3 for an explanation).

THEOREM 3. *Let $F \subset R$ and let k be fixed. Then there exists a continuous, linear extension operator E: $\Lambda^*_k(F) \to \Lambda_k(R)$. The norm of the operator depends only on k.*

Of course, the restriction operator from $\Lambda_k(R)$ to $\Lambda_k(F)$ is trivially continuous. The operator E in Theorem 3 is not the same for different values of k.

Next we give a characterization of $\Lambda_k(R)$ using interpolating polynomials, which is a little different than the one obtained by taking $F = R$ in Definition 3. For $k = 1$, the proposition below was given in [1], but the proof given there cannot be generalized to cover the cases $k > 1$. As before, a denotes a set consisting of $k + 1$ points from R.

PROPOSITION 1. *A function f belongs to $\Lambda_k(R)$ if and only if there exists a constant M such that $|f| \leqslant M$ and for every a*

$$|f(x) - P_a(x)| \leqslant M\,|a|^k, \qquad x \in Q_a. \tag{1.8}$$

The $\Lambda_k(R)$-norm of f is equivalent to the infimum of the constants M.

The proof of this proposition is given in Remark 1 in the next section.

2. PRELIMINARIES

This section is a preparation for the proof of Theorem 2, which is given in the next section. From now on, c denotes a constant, not necessarily the same each time it appears. Neither do we from time to time specify how c depends on other constants; in most formulas in this section, however, c is in a natural way equal to the $\Lambda_k^*(F)$-norm or the $\Lambda_k(F)$-norm of a given function f, multiplied by a constant which depends on k and maybe some other constants, which in their turn, when the formulas are applied in Section 3, will be chosen only depending on k. Lemma 1 and the first two remarks contain results on the space $\Lambda_k^*(F)$.

In the lemma, we use the same notation as in Definition 3, but we do not assume that a and b have k points in common, and furthermore lQ_a denotes the interval with the same center as Q_a but with length $l\,|a|$.

LEMMA 1. *Let $l \geqslant 1$, and let $f \in \Lambda_k^*(F)$. Then*

$$|P_a(x) - P_b(x)| \leqslant c(\max(|a|, |b|))^k, \qquad x \in lQ_a \cap lQ_b. \tag{2.1}$$

Proof. We may suppose that $|a| \leqslant |b|$. Assume first that a and b have k points in common, say $\alpha_1, \alpha_2, ..., \alpha_k$. Consider the zero-polynomial as the polynomial of degree at most $k-1$ interpolating $P_a - P_b$ at these points. Then, by Lagrange's interpolation formula with remainder, for some τ,

$$P_a(x) - P_b(x) - 0 = (P_a^{(k)}(\tau) - P_b^{(k)}(\tau)) \prod_{i=1}^{k} (x - \alpha_i)/k!.$$

Using (1.5) this gives

$$|P_a(x) - P_b(x)| \leqslant c(1 + \ln(|b|/|a|)) \prod_{i=1}^{k} |x - \alpha_i|. \tag{2.2}$$

If now $x \in lQ_a$, then $|x - \alpha_i| \leqslant l\,|a|$, and using also that $1 + \ln(|b|/|a|) \leqslant |b|/|a|$, we obtain from (2.2) that $|P_a(x) - P_b(x)| \leqslant c\,|b|\,|a|^{k-1} \leqslant c\,|b|^k$, which is (2.1) in case a and b have k points in common.

We next prove the lemma assuming that $Q_a \subset Q_b$. We do this by exchanging the elements of a by elements of b, one at a time, and using in each step the already proved case. Let $d_0 = a$. Let d_1 be as a, but with the smallest element of a replaced by the smallest in b. Let d_2 be as d_1 but with the largest element of d_1 replaced by the largest in b. Next let d_3 be as d_2, but with an element in d_2 which is not in b (if there is one) replaced by an element in b not in d_2; this last procedure is continued until we arrive at $d_v = b$. Here $v \leqslant k$. Then all Q_{d_v} contain Q_a, so from the special case of (2.1) shown above

$$|P_{d_i} - P_{d_{i-1}}| \leqslant c\,|b|^k, \qquad x \in lQ_{d_i} \cap lQ_{d_{i-1}} \supset lQ_a = lQ_a \cap lQ_b.$$

Using $|P_a - P_b| \leqslant \sum_{i=1}^{v} |P_{d_i} - P_{d_{i-1}}|$, we get (2.1) in the case $Q_a \subset Q_b$. Finally, the general case follows upon writing $|P_a - P_b| \leqslant |P_a - P_e| +$

$|P_e - P_b|$, where e is a set consisting of $k+1$ elements from $a \cup b$, among them the smallest and largest from $a \cup b$, so that $Q_a, Q_b \subset Q_e$.

Remark 1. If $f \in \Lambda_k^*(F)$, $x \in Q_a \cap F$, and P_x is a polynomial interpolating f at x and k points of a, then Lemma 1 shows that $|P_x(x) - P_a(x)| \leqslant c\,|a|^k$ so, since $P_x(x) = f(x)$,

$$|f(x) - P_a(x)| \leqslant c\,|a|^k, \qquad x \in Q_a \cap F. \tag{2.3}$$

In particular this shows, as soon as Theorem 2 is proved, that the only-if part of Proposition 1 holds. The converse follows easily from the characterization of $\Lambda_k(R)$ in Definition 2.

Remark 2. We shall deduce some more facts concerning $\Lambda_k^*(F)$ using Newton's interpolation formula. It may be written

$$P(a_0, a_1, ..., a_m; f)(x) = \sum_{i=0}^{m} P^{(i)}(a_0, a_1, ..., a_i; f) \prod_{j=0}^{i-1} (x - a_j)/i!, \tag{2.4}$$

where $P(a_0, a_1, ..., a_i; f)$ is the polynomial interpolating f at the distinct points $a_0, a_1, ..., a_i$. Combining (2.4) with (1.6) we obtain that if $f \in \Lambda_k^*(F)$ and Q has length $\leqslant c_1$,

$$|P_a(x)| \leqslant c(1 + \max(\ln(1/|a|), 0), \qquad x \in Q, \text{ if } Q_a \subset Q, \tag{2.5}$$

and

$$|P_{a(v)}(x)| \leqslant c, \qquad x \in Q, v < k, \text{ if } Q_{a(v)} \subset Q. \tag{2.6}$$

From (2.4) we also obtain the following result of a technical nature, which we will use in the next section. If $Q_a \subset Q$, where Q has diameter 1 and center x_0, and b is obtained from a by replacing the largest element γ of a by $\gamma + 1$, and P_b interpolates f at the points of b from a and to $f(\gamma)$ at $\gamma + 1$, then

$$|P_b^{(k)}| \leqslant c \tag{2.7}$$

if $f \in \Lambda_k^*(F)$. This follows if one applies (2.4) to P_b taking $x = a_m = \gamma + 1$ and using (1.6). We also obtain, again using (2.4), that $|P_b(x)| \leqslant c$, $x \in Q$ if $Q_b \subset Q$ where Q is as in (2.5).

The remaining remarks concern the space $\Lambda_k(F)$.

Remark 3. In the definition of $\Lambda_k(F)$ (Definition 2) we may as well assume that Q and Q' are centered in F, that $Q' \subset Q$, and that all cubes involved have sides of length $\leqslant 1$. In order to see this, assume that (1.2)–(1.4) hold with these restrictions on Q and Q'. First of all, one may assume that they hold for any Q, which is seen by setting $P_Q = 0$ if Q has sides of length >1 (note that, by (1.2)–(1.4), $|f(x)| \leqslant c$, $x \in F$, and $|P_{Q'}(x)| \leqslant c$, $x \in Q'$, $\delta' < 1$). If $Q \cap F \neq \varnothing$ but Q is not necessarily centered in F, let δ be the length of a side of Q. Take a cube R with center in F and sides, parallel to those of Q, of length 2δ; then $Q \subset R$. Define P_Q on Q by $P_Q = P_R$, where P_R is the polynomial associated to R by our assumption. Then, of course, P_Q satisfies (1.2) and (1.4), and we shall prove (1.3). Let Q and Q' intersect F and let $\delta' < \delta$. Let R'' be a cube with the same center as R containing both R and R', with sides of length not exceeding a fixed constant times δ. Then $|P_Q - P_{Q'}| = |P_R - P_{R'}| \leqslant |P_R - P_{R''}| + |P_{R''} - P_{R'}| \leqslant c\delta$ in $Q \cap Q'$.

Remark 4. Let $f \in \Lambda_k(F)$ where $F \subset R$, let Q, Q', P_Q, and $P_{Q'}$ be as in Definition 2 with $Q \cap Q' \neq \varnothing$, and assume that $\delta > \delta'$. Then

$$|P_Q^{(k)} - P_{Q'}^{(k)}| \leqslant c(1 + \ln(\delta/\delta')). \tag{2.8}$$

It is readily seen that it is enough to prove (2.8) in case $Q' \subset Q$ and Q and Q' have an endpoint x_0 in common (insert $\pm P_{Q \cup Q'}^{(k)}$ in $P_Q^{(k)} - P_{Q'}^{(k)}$ if $Q' \not\subset Q$). Let m be the first integer such that $e^{m}\delta' \geqslant \delta$, then $e^{m-1}\delta' < \delta$ so $m - 1 + \ln \delta' < \ln \delta$ or $m < 1 + \ln(\delta/\delta')$. Let Q_ν, $\nu = 0, 1, ..., m-1$, be the intervals with one endpoint x_0, containing Q', and of length $e^\nu \delta'$, and put $Q_m = Q$. Write $P_Q^{(k)} - P_{Q'}^{(k)} = \sum_{i=1}^m (P_{Q_i}^{(k)} - P_{Q_{i-1}}^{(k)})$. By (1.3) and Markov's inequality we have $|P_{Q_i}^{(k)} - P_{Q_{i-1}}^{(k)}| \leqslant c$, so we obtain (2.8). From (2.8) we may also obtain that

$$|P_{Q'}^{(k)}| \leqslant c(1 + \ln(1/\delta')), \qquad \delta' < 1, \tag{2.9}$$

by taking as Q an interval with length 1 containing Q'. Then $|P_Q^{(k)}| \leqslant c$ by (1.4) and Markov's inequality, and (2.9) follows upon writing $P_{Q'}^{(k)} = P_{Q'}^{(k)} - P_Q^{(k)} + P_Q^{(k)}$.

Remark 5. If $F = R$, the polynomials P_Q in Definition 2 may be chosen to satisfy not only (1.2), (1.3), and (1.4), but also (c_1 is a constant)

$$|f^{(j)} - P_Q^{(j)}| \leqslant c\delta^{k-|j|}, \qquad j < k, \quad x \in Q, \tag{2.10}$$

$$|P_Q^{(k)} - P_{Q'}^{(k)}| \leqslant c, \qquad \delta \leqslant c_1\delta', \quad Q' \subset Q, \tag{2.11}$$

and

$$|P_Q^{(j)}| \leqslant c, \qquad |j| \leqslant k-1, \quad \delta = 1. \tag{2.12}$$

An analogy of this holds if $F = R^n$, and actually, in a certain sense, if $F \subset R^n$. See [2].

3. Proof of Theorem 2

In the proof, we shall not say anything explicitly about equivalence of norms. However, all calculations are such that we actually obtain not only that $\Lambda_k(F) = \Lambda_k^*(F)$, but also that (1.7) holds.

We shall first prove that if $f \in \Lambda_k(F)$, then $f \in \Lambda_k^*(F)$. It is enough to prove this if $F = R$, since if $f \in \Lambda_k(F)$, then by Theorem 1 there exists an extension Ef of f to R, lying in $\Lambda_k(R)$. If this implies that $Ef \in \Lambda_k^*(R)$, then by the definition of the Λ_k^*-spaces, the restriction of Ef to F, i.e., f, belongs to $\Lambda_k^*(F)$.

So, let us assume that $f \in \Lambda_k(R)$, and let $a = \{a_0, a_1, ..., a_k\}$ and $b = \{a_0, a_1, ..., a_{k-1}, b_k\}$, where $a_0, a_1, ..., a_k$, and b_k are points from R such that $|b| \geqslant |a|$. We start by proving (1.5), i.e.,

$$|P_a^{(k)} - P_b^{(k)}| \leqslant c(1 + \ln(|b|/|a|)). \tag{3.1}$$

Recall that Q_a and Q_b are the smallest intervals containing a and b, respectively. Let P_{Q_a} and P_{Q_b} be polynomials associated to Q_a and Q_b, and to f, as in the definition of $\Lambda_k(F)$. We insert these in $P_a^{(k)} - P_b^{(k)}$ and obtain

$$|P_a^{(k)} - P_b^{(k)}| \leqslant |P_a^{(k)} - P_{Q_a}^{(k)}| + |P_{Q_a}^{(k)} - P_{Q_b}^{(k)}| + |P_{Q_b}^{(k)} - P_b^{(k)}|.$$

By (2.8), $|P_{Q_a}^{(k)} - P_{Q_b}^{(k)}| \leqslant c(1 + \ln(|b|/|a|))$. To estimate $P_a^{(k)} - P_{Q_a}^{(k)}$ we use that, since $P_a - P_{Q_a}$ interpolates $f - P_{Q_a}$ in $a_0, a_1, ..., a_k$, we may write (for $k > 1$; for $k = 1$ use the pointwise representation)

$$\begin{aligned} P_a^{(k)} - P_{Q_a}^{(k)} = \frac{k!}{a_k - a_0} \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{k-2}} \Bigg\{ & (f - P_{Q_a})^{(k-1)} \\ & \times \left(a_1 + \sum_{i=1}^{k-1} (a_{i+1} - a_i)\, t_i \right) - (f - P_{Q_a})^{(k-1)} \\ & \times \left(a_0 + \sum_{i=1}^{k-1} (a_i - a_{i-1})\, t_i \right) \Bigg\}\, dt_1, dt_2, ..., dt_{k-1}. \end{aligned} \tag{3.2}$$

This is a well-known representation (see, e.g., [3]; (3.2) may be obtained by combining the formulas (1.4) and (1.6) in that paper). By Remark 5 in Section 2, we may assume that $|f^{(k-1)}(x) - P_{Q_a}^{(k-1)}(x)| \leqslant c\,|a|$, $x \in Q_a$, and assuming for the moment that $a_0 \leqslant a_1 \leqslant \cdots \leqslant a_k$, we obtain from (3.2) that $|P_Q^{(k)} - P_{Q_a}^{(k)}| \leqslant c$; hence we also have $|P_b^{(k)} - P_{Q_b}^{(k)}| \leqslant c$. Altogether, we have proved (3.1).

Next we shall see that (1.6) holds. Since $|f^{(k-1)}| \leqslant c$ it follows from (3.2) that $|P_a^{(k)}| \leqslant c$ if $|a| > 1$ (use (3.2) with $f - P_{Q_a}$ replaced by f and $P_a^{(k)} - P_{Q_a}^{(k)}$ replaced by $P_a^{(k)}$). If $|a| < 1$ we write $|P_a^{(k)}| \leqslant |P_a^{(k)} - P_{Q_a}^{(k)}| + |P_{Q_a}^{(k)}|$; then by what we proved above $|P_a^{(k)} - P_{Q_a}^{(k)}| \leqslant c$, and by (2.9) $|P_{Q_a}^{(k)}| \leqslant c(1 +$

$\ln(1/|a|))$. Thus we have shown that $|P_a^{(k)}| \leqslant c(1+\max(0, \ln(1/|a|)))$. Of course $|P_{a(0)}^{(0)}(x)| = |f(x)| \leqslant c$, and $|P_{a(v)}^{(v)}| \leqslant c$ may be obtained immediately from formula (1.6) in [3] and the fact that $|f^{(j)}| \leqslant c$, $|j| < k$.

In the proof of the converse part of Theorem 2, we assume that F contains at least $k+1$ points. This is permitted since a function in $\Lambda_k^*(F)$ is bounded, by (1.6), and it is easy to see that a bounded function on a finite set F is in $\Lambda_k(F)$. We need the following construction, essentially given in [7]. Let $x_0 \in F$ be an isolated point of F. We associate to x_0 a set $Q(x_0) = \{a_0, a_1, ..., a_k\}$ consisting of $k+1$ distinct points from F in the following way.

Let $a_0 = x_0$. Let a_1 be the element of F closest to a_0 (if there are two, take the one to the right), let a_2 be the point of F closest to $\{a_0, a_1\}$ (again, if two, take the one to the right), let a_3 be the point of F closest to $\{a_0, a_1, a_2\}$, and so on. We continue this procedure until we arrive at a limit point of F, or until $k+1$ points are chosen. If we arrive at a limit point a_i, then choose $a_{i+1}, ..., a_k$ from F according to some rule so that they are closer to a_i then any of the points $a_0, a_1, ..., a_{i-1}$.

Let $d(Q(x_0))$ denote the diameter of $Q(x_0)$. We need the following fact about this construction (cf. [3, Lemma 3.4] and [7, Lemma 8]).

LEMMA 2. *Let x_0 be an isolated point of F, and let $y_0 \in F$. Suppose that $d(Q(x_0)) > k\,|x_0 - y_0|$. Then y_0 is isolated, and $Q(x_0) = Q(y_0)$.*

Proof. Let v be the first integer such that the distance from a_{v+1} to $\{a_0, a_1, ..., a_v\} = S_v$ is greater than $|x_0 - y_0|$. The integer v exists since $d(Q(x_0)) > k\,|x_0 - y_0|$. Then $a_0, a_1, ..., a_v$ are isolated, and y_0 must be one of them. Furthermore, if we enumerate the points of S_v from left to right, the distance between two consecutive points is less than or equal to $|x_0 - y_0|$. But this means that when constructing $Q(y_0)$, the first $v+1$ points will be those of S_v, which means that $Q(x_0) = Q(y_0)$.

Assume now that $f \in \Lambda_k^*(F)$. We shall prove that $f \in \Lambda_k(F)$ by defining, for each interval Q of length $\leqslant 1$ centered in F (cf. Remark 3) a polynomial P_Q satisfying (1.2), (1.3), and (1.4) in Definition 2. Let Q have length δ and center $x_0 \in F$. When defining P_Q, we consider two different cases. Case 1: Q contains at least $k+1$ points from F. Case 2: Q contains at most k points from F.

In Case 1, we define P_Q in the following way. Let α be the point of $F \cap Q$ furthest away from x_0 (if there are two, take the one to the right). Put $a_0 = \alpha$. Let $a_1 \in F \cap Q$ be the point furthest away from a_0, and let, for $i = 2, 3, ..., k$, $a_i \in F \cap Q$ be furthest away from $\{a_0, a_1, ..., a_{i-1}\}$. If there are several possibilities, take the one furthest to the right; however, it is not important how $a_2, a_3, ..., a_k$ are chosen, as long as they are chosen according to some rule. Put $a = \{a_0, a_1, ..., a_k\}$. Next, let β be the point in F, but not in the interior of Q, closest to Q_a, if there is such a point with distance $\leqslant 1$ from a. If there are two, take the one to the right, and if there are none, put $\beta=\gamma+1$, where γ is the largest element of $F\cap Q$. Let b be as a, but with a_0 or a_1 replaced by β, in such a way that $Q_a\subset Q_b$, and let P_a and P_b be the polynomials of degree $\leqslant k$ interpolating to f at a and b, respectively. If $\beta=\gamma+1$, let instead P_b have the value $f(\gamma)$ at β. By (1.6) and (2.7) we have $|P_a^{(k)}-P_b^{(k)}|\leqslant c(1+\ln(1/|a|))+c\leqslant c(1+\ln|b|/|a|)$. As a consequence, the formula (2.2) which we use below is valid also in this case, which will mean that we do not have to treat it separately. If $a=b$, put $\theta=1$; otherwise let θ be given by

$$\theta\ln|a|+(1-\theta)\ln|b|=\ln r, \tag{3.3}$$

where r is the length of $Q\cap Q_b$.

We define P_Q in Case 1 by

$$P_Q=\theta P_a+(1-\theta)P_b. \tag{3.4}$$

For future reference, we note that by (3.3) we have, if $a\neq b$,

$$\theta=\ln(|b|/r)/\ln(|b|/|a|) \tag{3.5}$$

and

$$1-\theta=\ln(r/|a|)/\ln(|b|/|a|). \tag{3.6}$$

Also, if $x\in Q$ then by (2.2) we have $|P_a-P_b|\leqslant c\delta^k(1+\ln(|b|/|a|))$, so

$$|P_a(x)-P_b(x)|\leqslant c\delta^k, \qquad x\in Q, \quad |b|\leqslant 2|a|. \tag{3.7}$$

In Case 2, associate $Q(x_0)$ to x_0 as in Lemma 2. If $d(Q(x_0))>k/2$, let $a_{\mu+1}$ be the first point in the construction of $Q(x_0)$ such that the distance from $a_{\mu+1}$ to $T(x_0)=\{a_0,a_1,\ldots,a_\mu\}$ is $>\frac{1}{2}$. Put $d=Q(x_0)$ and $d(\mu)=T(x_0)$, let P_d and $P_{d(\mu)}$ be polynomials of degree $\leqslant k$ and $\leqslant\mu$, respectively, interpolating to f at the points of d and $d(\mu)$, and define P_Q by

$$P_Q=P_d \qquad \text{if} \quad d(Q(x_0))\leqslant k/2 \tag{3.8}$$

and

$$P_Q=P_{d(\mu)} \qquad \text{otherwise}. \tag{3.9}$$

For future reference, we note that if $d(Q(x_0))>k/2$ and $|x_0-y_0|<\frac{1}{2}$, then $T(x_0)=T(y_0)$. This follows from the proof of Lemma 2, which shows that for some $\nu\leqslant\mu$, the first ν points chosen in the definition of $Q(x_0)$ form the same set as the first ν chosen in the definition of $Q(y_0)$.

We shall now show that (1.2)–(1.4) hold under the assumptions that both Q and Q' are centered in F, $Q' \subset Q$, and $\delta \leqslant 1$ (see Remark 3), and start with (1.3), i.e., $|P_Q - P_{Q'}| \leqslant c\delta^k$, $x \in Q \cap Q'$. Notationally, we put a prime on concepts related to Q'; the center of Q' is for example denoted by x_0'. Three cases may occur: (i) Both P_Q and $P_{Q'}$ are defined by Case 1; (ii) P_Q is given by Case 1 but $P_{Q'}$ by Case 2, and $P_{Q'}$ is defined by means of (3.8); or (iii) both P_Q and $P_{Q'}$ are given by Case 2. We treat the cases (i), (ii), and (iii) separately.

In case (i) we have

$$P_Q - P_{Q'} = \theta P_a + (1-\theta) P_b - \theta' P_{a'} - (1-\theta') P_{b'}.$$

Assume first that the interior of Q', $\mathrm{Int}(Q')$, contains $Q \cap F$. Then $a = a'$ and $b = b'$, so $P_Q - P_{Q'} = (\theta - \theta')(P_a - P_b)$. Using (2.2) for $x \in Q'$ (then $|x - a_i| \leqslant \delta'$) and (3.5) one obtains (use $\delta/2 \leqslant r \leqslant \delta$)

$$|(\theta - \theta')(P_a - P_b)| \leqslant c \ln(2\delta/\delta')/\ln(|b|/|a|)\, \delta'^k (1 + \ln(|b|/|a|)).$$

Thus, if $|b| \geqslant 2\,|a|$, since $\delta'^k \ln(\delta/\delta') \leqslant \delta^k$, we have $|P_Q - P_{Q'}| \leqslant c\delta^k$, $x \in Q'$. If $|b| \leqslant 2\,|a|$, this estimate follows directly from (3.7). In case (i), when $\mathrm{Int}(Q') \not\supset Q \cap F$, we instead write

$$P_Q - P_{Q'} = (1-\theta)(P_b - P_a) + \theta'(P_{b'} - P_{a'}) + P_a - P_{b'}.$$

Since x_0', the center of Q', is in Q_a and $\mathrm{Int}(Q') \not\supset Q \cap F$, it is easily seen that $3Q_a \supset Q'$, so $|x - a_i| \leqslant 2\,|a|$ if $x \in Q'$. With this in mind, one obtains for $x \in Q'$ from (2.2) and (3.6) that $(1-\theta)\,|P_b - P_a| \leqslant c \ln(\delta/|a|)/\ln(|b|/|a|)$ $|a|^k (1 + \ln(|b|/|a|)) \leqslant c\delta^k$ if $|b| \geqslant 2\,|a|$. If not, we use instead (3.7). Next, by (2.2) and (3.5) we obtain for $x \in Q'$, using the fact that $r' \geqslant \delta'/2$, $\theta'\,|P_{b'} - P_{a'}| \leqslant c \ln(2\,|b'|/\delta')/\ln(|b'|/|a'|)\delta'^k(1 + \ln(|b'|/|a'|)) \leqslant c\delta'^k$ $\ln(|b'|/\delta')$ if $|b'| \geqslant 2\,|a'|$ (if not, we again estimate by means of (3.7)). Since $\mathrm{Int}(Q') \not\supset Q \cap F$, we have $|b'| \leqslant \delta$, and hence $\theta'\,|P_{b'} - P_{a'}| \leqslant c\delta^k$. Finally, since it is easily seen that $3Q_a \cap 3Q_{b'}$ contains Q' if $\mathrm{Int}(Q') \not\supset Q \cap F$, Lemma 1 gives $|P_a - P_{b'}| \leqslant c\delta^k$, $x \in Q'$. This completes the proof of (1.3) in case (i).

In case (ii), $P_Q - P_{Q'} = \theta P_a + (1-\theta) P_b - P_{d'} = (1-\theta)(P_b - P_a) + P_a - P_{d'}$. Since now $Q' \not\supset Q \cap F$, the calculations above give $(1-\theta)$ $|P_b - P_a| \leqslant c\delta^k$, $x \in Q'$. It is easily seen that, by construction, $|d'| \leqslant (k/2)\,\delta$, so by Lemma 1, $|P_a - P_{d'}| \leqslant c\delta^k$, $x \in 3Q_a \cap 3Q_{d'} \supset Q'$. Note that $3Q_{d'} \supset Q'$ follows from the fact that $Q_{d'}$ contains points outside Q' for if not Q' would contain $k+1$ points.

Finally we consider case (iii). If $d(Q(x_0)) > (k/2)\,\delta$, then, since $|x_0 - x_0'| \leqslant \delta/2$, it follows from Lemma 2 that $Q(x_0) = Q(x_0')$. If furthermore $d(Q(x_0)) > k/2$, then, by the comments after (3.9), $T(x_0) = T(x_0')$. Thus, by the definition of P_Q and $P_{Q'}$ in Case 2, $P_Q = P_{Q'}$. On the other hand, if $d(Q(x_0)) \leqslant (k/2)\,\delta$, then also $d(Q(x_0')) \leqslant (k/2)\,\delta$, both P_Q and $P_{Q'}$ are defined using (3.8), and Lemma 1 gives $|P_Q - P_{Q'}| = |P_d - P_{d'}| \leqslant c\delta^k$, $x \in Q' \subset 3Q_d \cap 3Q_{d'}$. With this, (1.3) is proved also in case (iii).

The proof of (1.2) is now obtained from (1.3) as follows. If x is a cluster point of $Q \cap F$, let Q' be a subinterval of Q with center in F and one endpoint x, containing at least $k+1$ points from F. Then $f(x) = P_{Q'}(x)$, so, by (1.3), $|f(x) - P_Q(x)| = |P_{Q'}(x) - P_Q(x)| \leqslant c\delta^k$. On the other hand, if x is an isolated point of $Q \cap F$, consider, if $x \neq x_0$ ($x = x_0$ yields $f(x) = P_Q(x)$), an interval $Q'' \subset 2Q$ centered at x containing only x from $Q \cap F$. Then $f(x) = P_{Q''}(x)$, and since $Q'' \subset 2Q$ we get

$$|f(x) - P_Q(x)| \leqslant |P_{Q''}(x) - P_{2Q}(x)| + |P_{2Q}(x) - P_Q(x)| \leqslant c\delta^k$$

if $2Q$ has length $\leqslant 1$; otherwise (1.2) follows easily from (1.6), (1.3), and (1.4), which is proved below.

Finally we prove (1.4). If P_Q is defined by Case 1, then $|P_Q| \leqslant \theta\,|P_a| + (1-\theta)\,|P_b|$. Here, $|P_b| \leqslant c$ and $|P_a| \leqslant c$ if $|a| \geqslant \frac{1}{4}$ (see (2.5) and the discussion in the end of Remark 2). If $|a| \leqslant \frac{1}{4}$ we get by (3.5) and (2.5) that

$$\theta\,|P_a| \leqslant c \ln |b|/\ln(|b|/|a|)(1 + \ln(|b|/|a|)) \leqslant c.$$

On the other hand, if P_Q is defined by Case 2, then $P_Q = P_d$ where $1/2 \leqslant |d| \leqslant k/2$ so $|P| \leqslant c$ by (2.5), or $P_Q = P_{d(\mu)}$ where $d(\mu)$ has length $\leqslant k/2$, so $|P_Q| \leqslant c$ because of (2.6). This completes the proof of Theorem 2.

Now, we also obtain Theorem 3. By Theorem 1 and the comments after it, every $f \in \Lambda_k(F)$ has an extension Ef in $\Lambda_k(R^n)$ with $\|Ef\|_{\Lambda_k(R^n)} \leqslant c\,\|f\|_{\Lambda_k(F)}$. The extension Ef is constructed, e.g., with aid of the approximating polynomials $\{P_Q\}$ in Definition 2 (see [1, 2, 5]: however, the construction given there is not explicitly in terms of $\{P_Q\}$; combine, for $k=1$, e.g., the extension given on p. 168 in [1] with Remark 4.5 in that paper). If $\{P_Q\}$ is associated to f and $\{\tilde{P}_Q\}$ to $\tilde{f}$, and Ef and $E\tilde{f}$ are obtained by means of these approximations, then $Ef + E\tilde{f}$ is equal to $E(f+\tilde{f})$, if $E(f+\tilde{f})$ is obtained by means of the approximation $\{P_Q + \tilde{P}_Q\}$ to $f+\tilde{f}$. Now, if $F \subset R$ and $f \in \Lambda_k(F) = \Lambda_k^*(F)$, associate P_Q to f as in the proof of the statement ($f \in \Lambda_k^*(F) \Rightarrow f \in \Lambda_k(F)$) above. Then, by construction, $\{P_Q\}$ is associated to f in a linear way, and if Ef is constructed by means of this approximation, one obtains Theorem 3.

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